Assumptionless bounds for random trees

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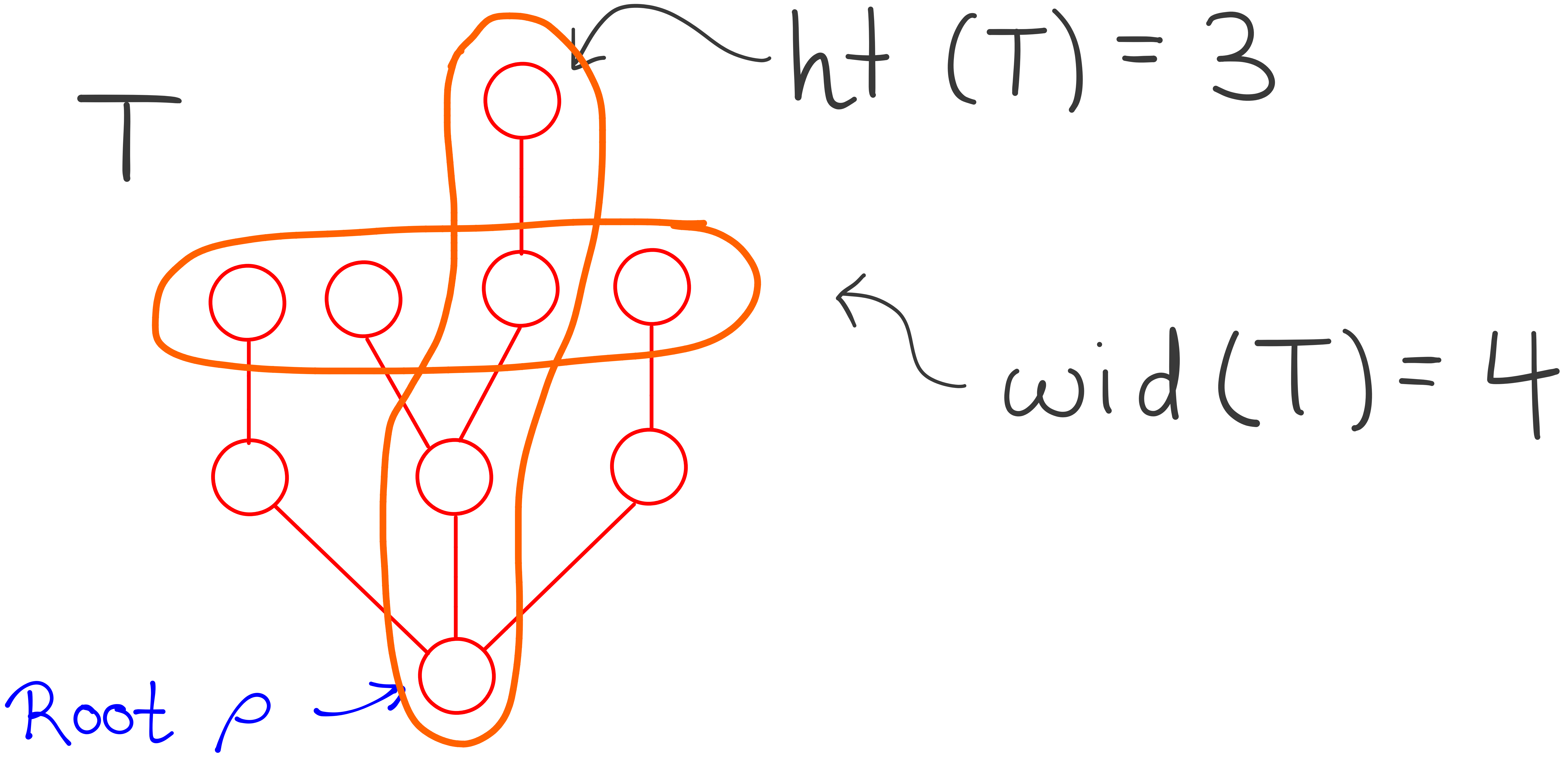
Trees:



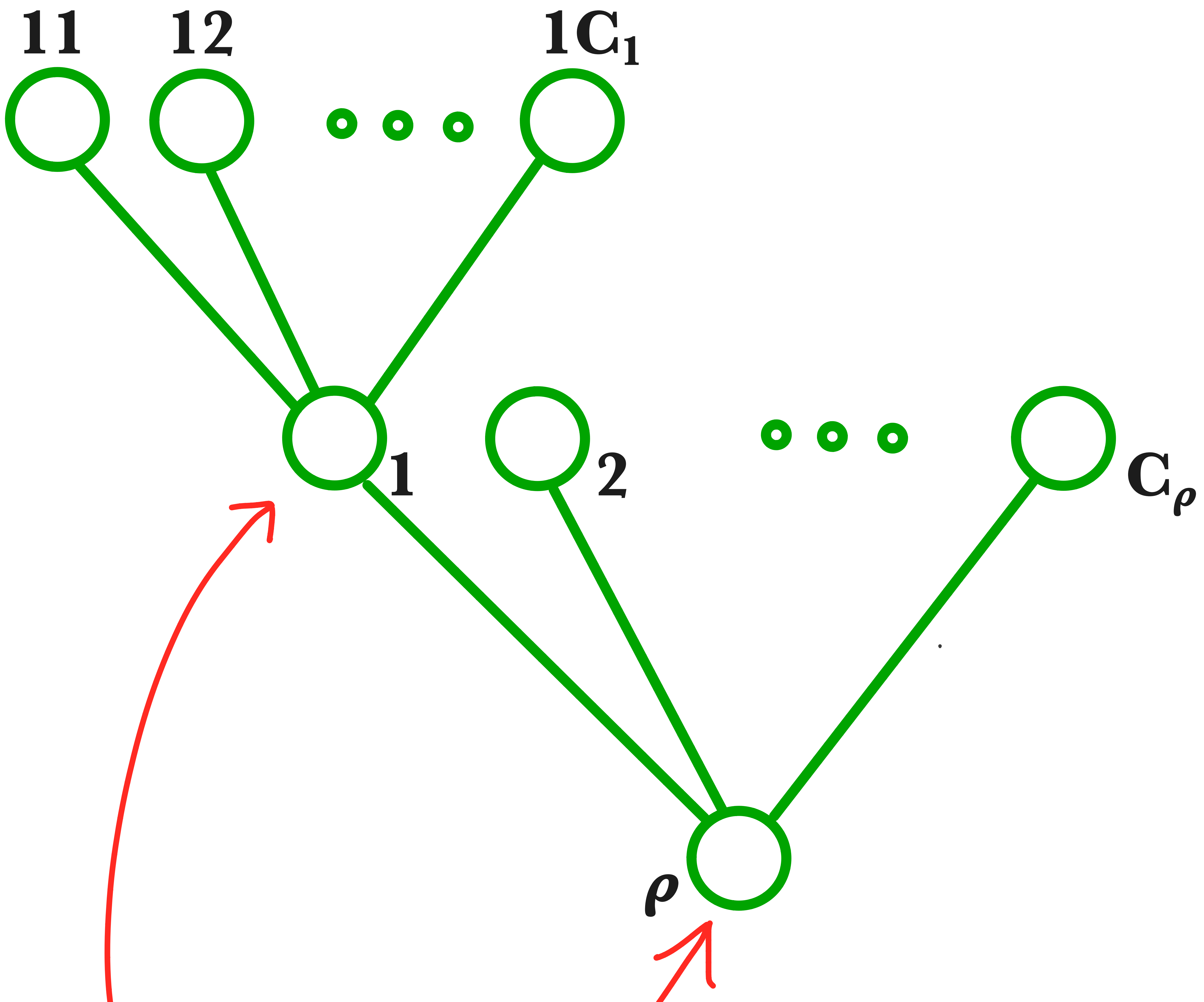
Height: Greatest distance from any node to the root } ht(T)

Width: Greatest # nodes on a single level. } wid(T)

Main Results

Fix any prob. distribution $p = (p_0, p_1, p_2, \ldots)$ on $\mathbb{N}$ (so $\sum_{i \geq 0} p_i = 1$)

Let T be $GW(p)$ distributed.



\# children is distributed as p, independently at each node.

Main Results

Fix any prob. distribution $p=(p_0,p_1,p_2,\ldots)$ on $\mathbb{N}$ (so $\sum_{i\geq 0} p_i = 1$)

Let T be $GW(p)$ distributed. Write $\mu(p) = \sum_{i\geq 0} i p_i \in [0,\infty]$.

Theorem ("Most trees are short & fat")

There is a universal constant $\delta > 0$ s.t.

$$\mathbb{P}\left(ht(T) > \frac{k}{1-p_1}\cdot wid(T)\right) \leq \exp(-\delta k).$$

Remark: Let $\sigma = \#$ nodes of T. If $\mu(p) > 1$ then $\mathbb{P}(\sigma=\infty) > 0$, and

$$\mathbb{P}(ht(T) = wid(T) = \infty \mid \sigma = \infty) = 1.$$

Also, given that $\sigma < \infty$, the cond. dist. of T is $GW(\hat{p})$ where $\hat{p}_1 = p_1$,

$\mu(\hat{p}) \leq 1$ so can assume $\mu(p) \leq 1$.

Heuristic: GW trees satisfy $wid(T)\cdot ht(T) \simeq \sigma$

Implies "$ht > C^2 \cdot wid$" $\hat{=}$ "$ht^2 \geq C^2 \cdot \sigma$"

So might guess that $\mathbb{P}\left(ht(T) > \frac{k}{\sqrt{1-p_1}}\sqrt{\sigma}\right) \leq \exp(-\delta k^2)$

Main Results

Fix any prob. distribution $p=(p_0,p_1,p_2,\dots)$ on $\mathbb{N}$ (so $\sum_{i\geq 0} p_i = 1$)

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Heuristic: GW trees satisfy $wid(T)\cdot ht(T) \cong \sigma$

Implies "$ht > C^2 \cdot wid$" $\hat{=}$ "$ht^2 \geq C^2 \cdot \sigma$" so $\mathbb{P}\left(ht(T) > \frac{k}{\sqrt{1-p_1}}\sqrt{\sigma}\right) \leq \exp(-\delta k^2)$

Theorem: $\mathbb{P}\left(ht(T) > \frac{k}{\sqrt{1-p_1}}\,\sigma^{1/2}\right) \leq \exp\left(-\delta k^2\right)$

Heuristic: GW trees satisfy $\mathrm{wid}(T)\cdot\mathrm{ht}(T) \cong \sigma$

Question: What is $\sup \mathbb{E}[\mathrm{wid}(T)\cdot\mathrm{ht}(T)/\sigma]$?

Supremum over p where $T \sim GW(p)$.

Question: What is the behaviour of

$$\sup \mathbb{E}[\mathrm{wid}(T)\cdot\mathrm{ht}(T)/\sigma \mid \sigma = n]$$

as a function of n?

(At least $c\log n$; lower bound in article)

Galton-Watson Trees

- Each node has random # of children
- Nodes reproduce independently

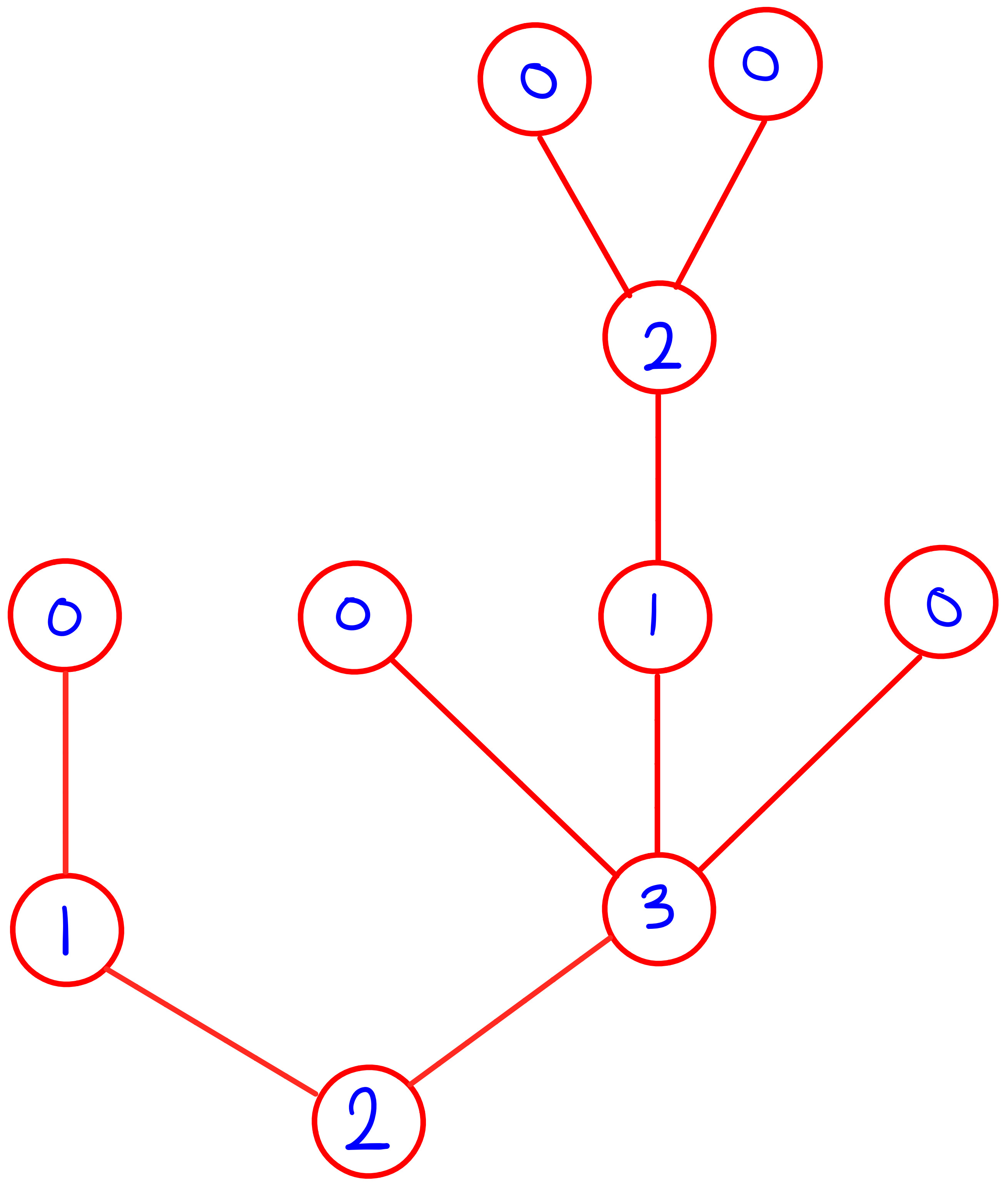
Construction

- $(C_i, i \geq 1)$ independent copies of a random variable C with $\sum_{k \geq 0} \mathbb{P}(C = k) = 1$.

Rule

The sequence $(C_i, i \geq 1)$ gives # children of nodes, in breadth-first search order

Example: $(2, 1, 3, 0, 0, 1, 0, 2, 0, 0, 1, 4, 0, \dots)$



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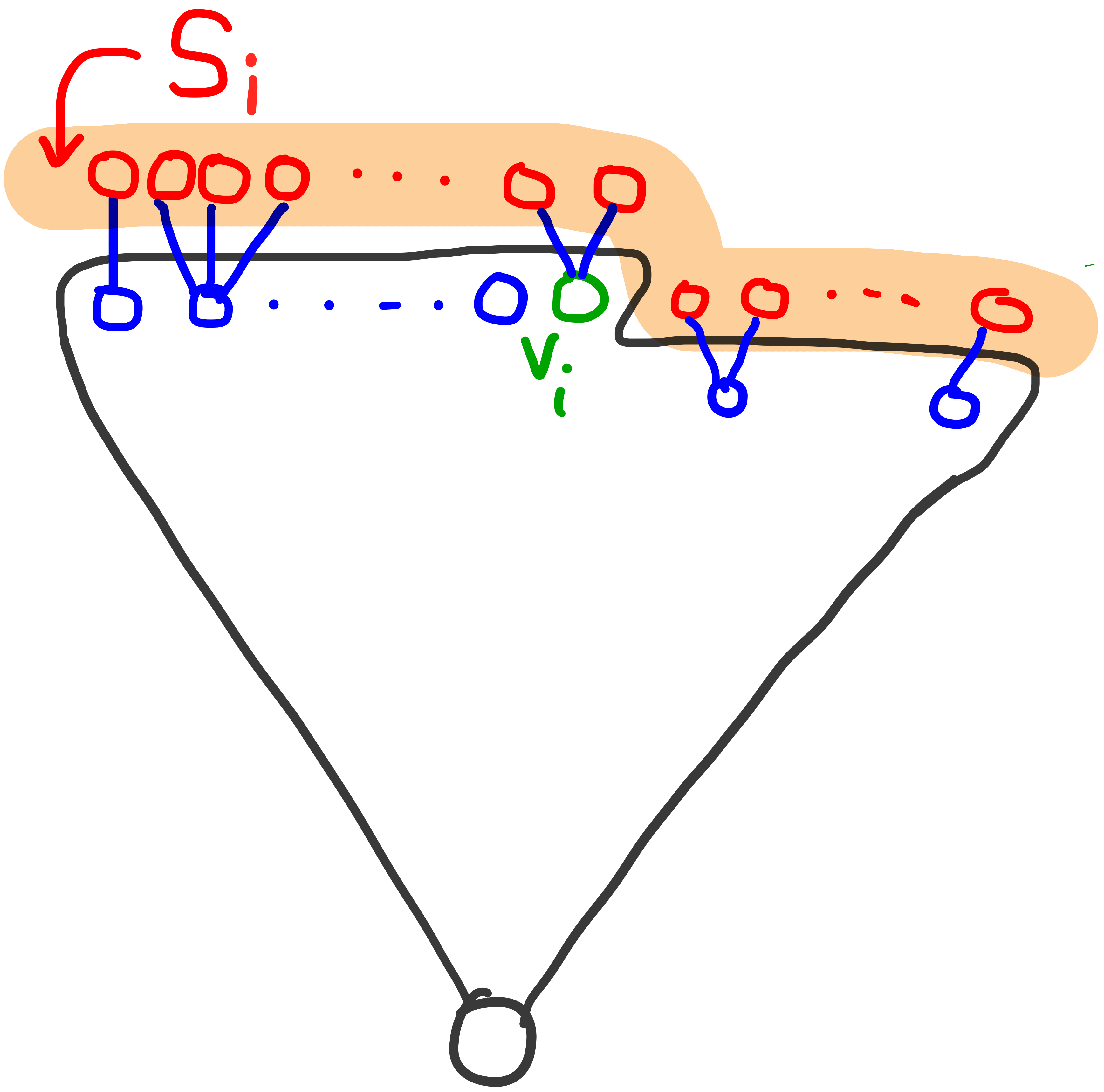


Halting Condition

For $i \geq 0$

#nodes discovered by $\overbrace{\text{time } i}^{i \text{ steps of BFS}}$.

$$= 1 + \sum_{j=1}^{i} C_j$$

$\#$ vertices explored by time $i = i$

Let $S_i = 1 + \sum_{j=1}^{i} (C_j - 1)$

= # nodes in "BFS queue" at time i

(discovered but not explored)

$S = (S_i, i \geq 0)$ = BFS queue process.

(determines T)

Total # vertices = σ

$$= \inf\left\{n : 1 + \sum_{j=1}^{n} C_j = n\right\} = \inf\left\{n : S_n = 0\right\}$$

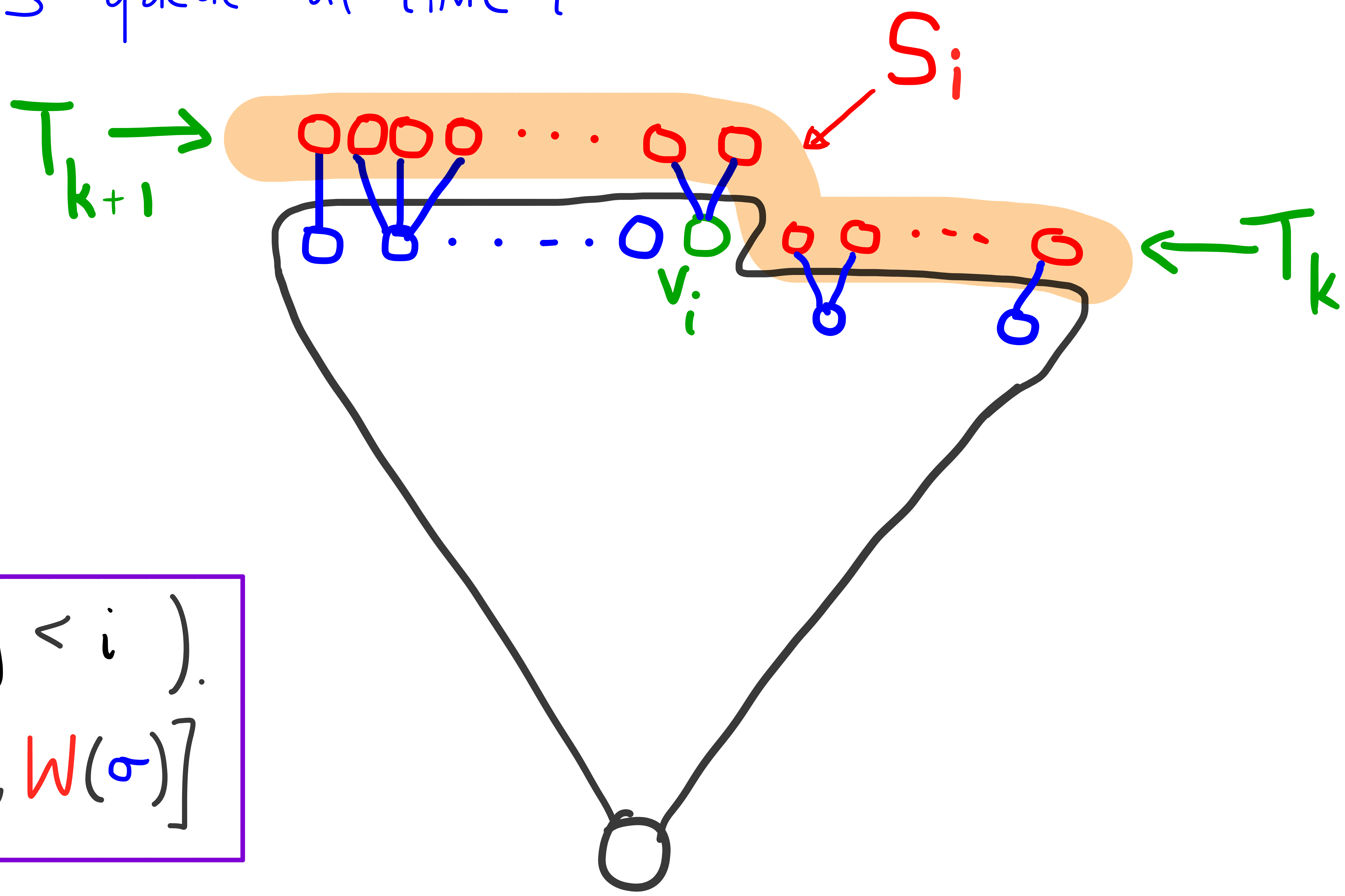
Setup

Let $S_i = 1 + \sum_{j=1}^{i} (C_j - 1)$ = # nodes in "BFS queue" at time i

$\mathbb{E}\, C \le 1 \implies \mathbb{E}(S_{n+1} - S_n) = \mathbb{E}\, C - 1 \le 0$

$\sigma = \inf\{ n : S_n = 0 \}$

= first time no nodes left to explore



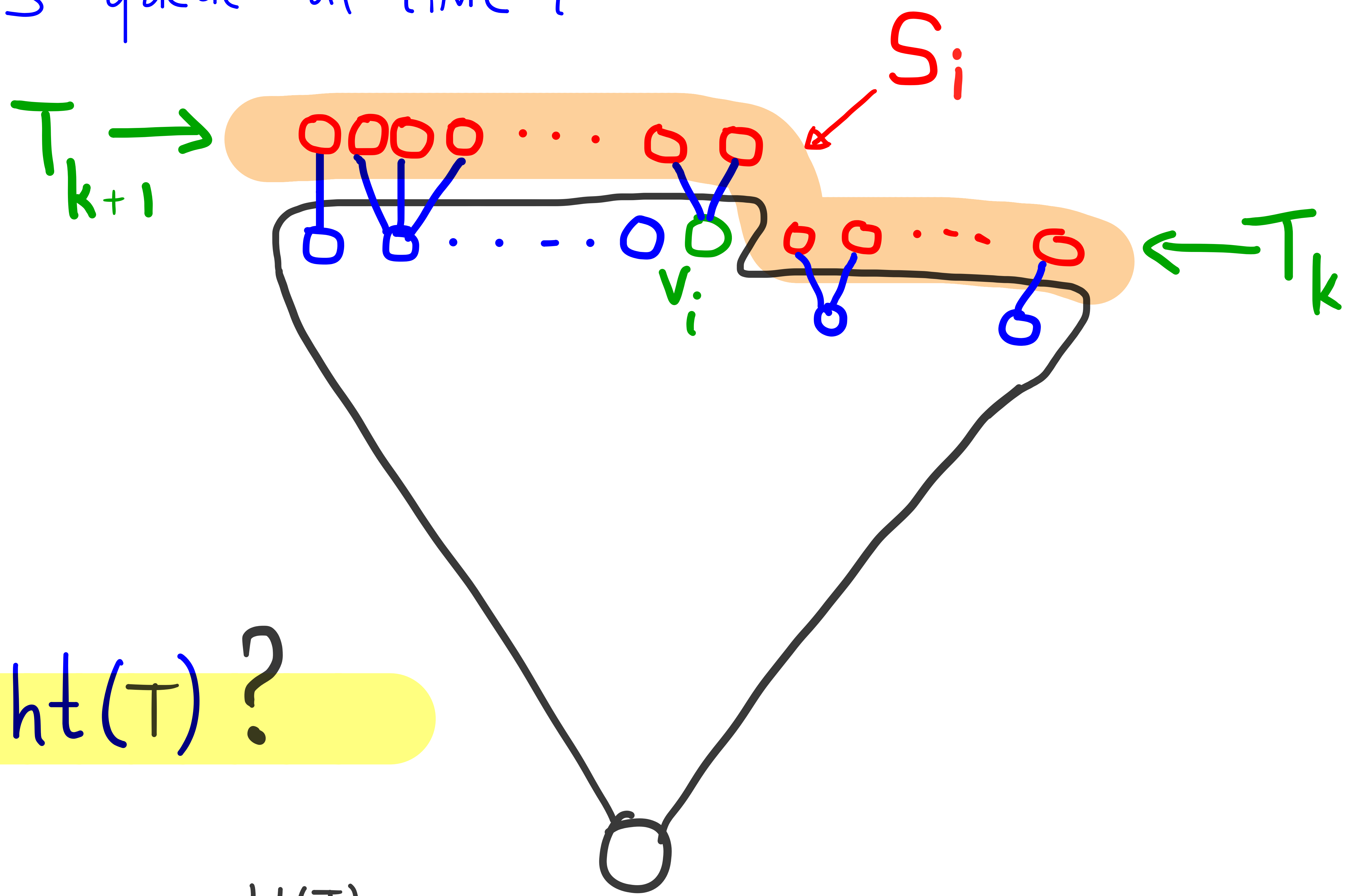
Prop: Let $W(i) = \max(S_j, 0 \le j < i)$.
Then $\mathrm{wid}(T) \in (W(\sigma)/2, W(\sigma)]$

Proof: During BFS on level k, "exploration queue" $\subset T_k \cup T_{k+1}$; and $= T_k$ at start of level k. ■

So $W(\sigma)$ is a good proxy for $\mathrm{wid}(T)$.

Setup

Let $S_i = 1 + \sum_{j=1}^{i}(C_j - 1)$ = # nodes in "BFS queue" at time i



$\mathbb{E}C \le 1 \implies \mathbb{E}(S_{n+1} - S_n) = \mathbb{E}C - 1 \le 0$

$\sigma = \inf\{n : S_n = 0\}$
= first time no nodes left to explore

What can act as a proxy for ht(T)?

Idea: $ht(T) = \sum_{k=1}^{ht(T)} 1 = \sum_{k=1}^{ht(T)} \sum_{v \in T_k} \frac{1}{|T_k|} \approx \sum_{k=1}^{ht(T)} \underbrace{\sum_{v_i \in T_k} \frac{1}{S_i}} = \sum_{i=1}^{\sigma} \frac{1}{S_i} =: H(\sigma)$

When $v_i \in T_k$ then $S_i \approx |T_k|$

Prop:
$ht(T) \le 3H(\sigma)$.

Proof: If $v_i \in T_k$ then $S_i \le |T_k \cup T_{k+1}|$, so $\sum_{v_i \in T_k} \frac{1}{S_i} \ge \frac{|T_k|}{|T_k \cup T_{k+1}|}$

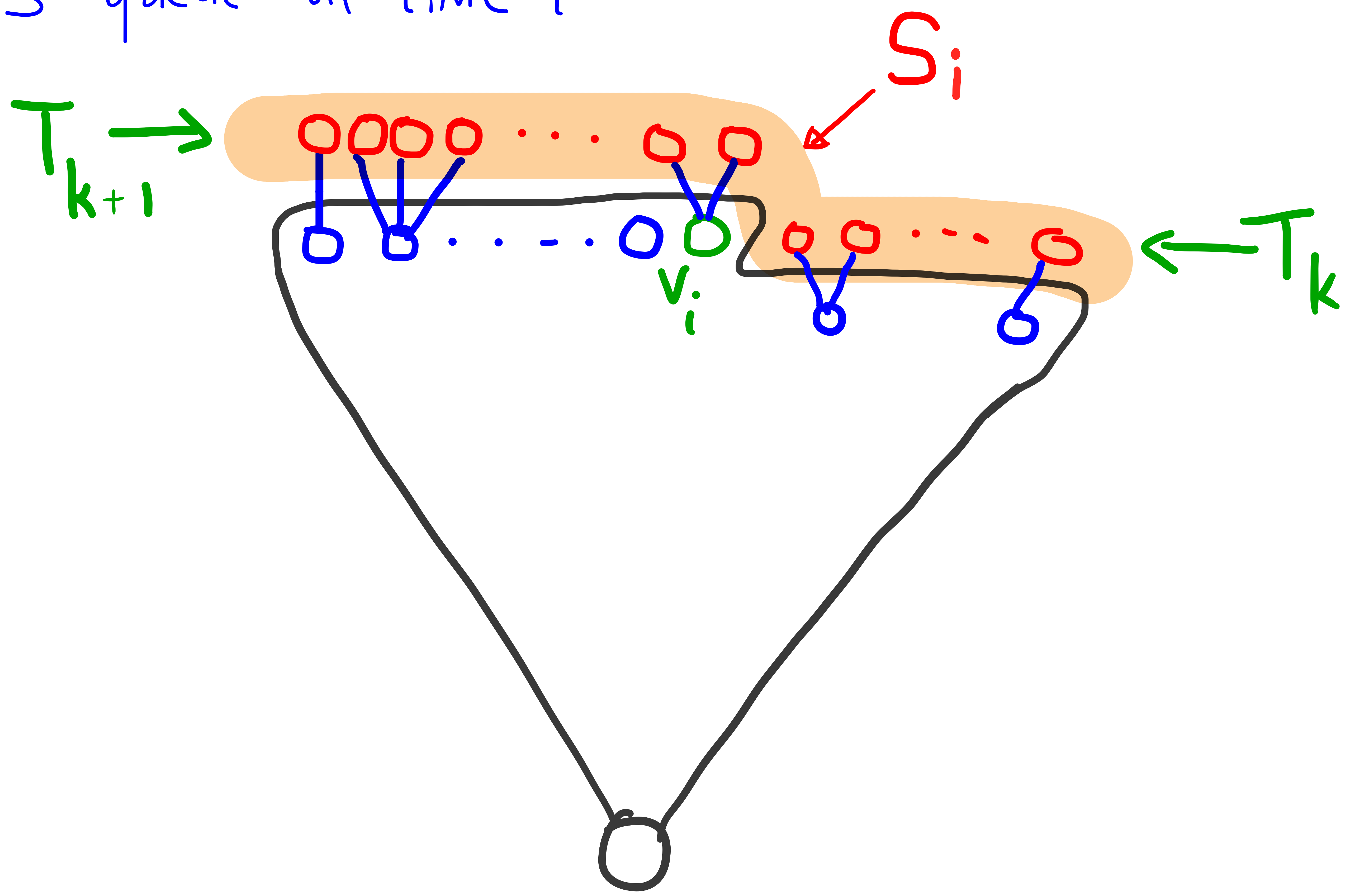
If v_i is the j'th node in T_k then $S_i \le |T_k \cup T_{k+1}| - j$ so $\sum_{v_i \in T_k} \frac{1}{S_i} \ge \sum_{j \le |T_k|} \frac{1}{|T_k \cup T_{k+1}| - j}$

$\ge \log\left(\frac{|T_k \cup T_{k+1}|}{|T_{k+1}|}\right)$

Combine these bounds, use that $S_0 = 1$, $S_\sigma = 0$. ∎

Setup

Let $S_i = 1 + \sum_{j=1}^{i}(C_j - 1) = \#$ nodes in "BFS queue" at time i



$\mathbb{E}C \le 1 \implies \mathbb{E}(S_{n+1} - S_n) = \mathbb{E}C - 1 \le 0$

$\sigma = \inf\{n : S_n = 0\}$

= first time no nodes left to explore

$$W(i) = \max(S_j, 0 \le j < i).$$

$$H(i) = \sum_{j=1}^{i} \frac{1}{S_j}$$

Prop: $\mathrm{wid}(T) \in (W(\sigma)/2, W(\sigma)]$

Prop: $\mathrm{ht}(T) \le 3H(\sigma).$

Corollary To prove $\mathbb{P}\left(\mathrm{ht}(T) \ge \frac{k}{1-p_1} \cdot \mathrm{wid}(T)\right) \le \exp(-\delta k)$

suffices to prove $\mathbb{P}\left(H(\sigma) \ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}.$

$W(i)=\max(S_j, 0\le j<i)$ $H(\sigma)=\sum_{j=1} \frac{1}{S_j}$ **Aim**: $\mathbb{P}\left(H(\sigma)\ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j \approx 2^\ell$ ("scale ℓ"), have $H(j)-H(j-1) = \frac{1}{S_j} \simeq \frac{1}{2^\ell}$

So bound (a) time to change scales,

(b) "# visits to scales" $=(V(\ell), \ell\ge 1)$

(a) Thm (Lévy; Doeblin; Kolmogorov; Rogozin; Le Cam; Esséen; Kesten):

With $p=\max p_i$, have $\boxed{\max_k \mathbb{P}(S_n=k) \le \frac{Cp}{\sqrt{n(1-p)}}}$ $C>0$ universal.

"Any random walk spreads out over $\ge \sqrt{n}$ values by time n".

Here $\sqrt{n} \simeq 2^\ell$.

Thus if $S_i\in(2^\ell, 2^{\ell+1}]$, $\tau=\inf\{j>i: S_j \notin (2^{\ell-1}, 2^{\ell+2}]\}$

then $\frac{\tau-i}{4^\ell}$ has subexponential upper tail: $\mathbb{P}(\tau-i\ge k\cdot 4^\ell)\le e^{-\delta k}$.

So $\frac{H(\tau)-H(i)}{2^\ell} \le \frac{(\tau-i)/2^{\ell-1}}{2^\ell} = 2\cdot\frac{\tau-i}{4^\ell}$ also has subexponential tail.

$W(\sigma) = \max(S_i, 0 \le i < \sigma)$ $H(\sigma) = \sum_{i=1}^{\sigma} \frac{1}{S_i}$ Aim: $\mathbb{P}\left(H(\sigma) \ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j \approx 2^\ell$ ("scale ℓ"), have $H(j) - H(j-1) = \frac{1}{S_j} \simeq \frac{1}{2^\ell}$

So bound (a) time to change scales,

(b) "# visits to scales" $= (V(\ell), \ell \ge 1)$

(b) $V(\ell) =$ # visits to scale ℓ

A visit to scale ℓ starts at a time i with $S_i \in (2^\ell, 2^{\ell+1}]$;

ends at time $\tau = \inf\{j > i : S_j \notin (2^{\ell-1}, 2^{\ell+2}]\}$

$W(\sigma)=\max(S_i, 0\le i<\sigma)$ $H(\sigma)=\sum_{i=1}^{\sigma}\frac{1}{S_i}$ Aim: $\mathbb{P}\left(H(\sigma)\ge\frac{k}{1-p_1}W(\sigma)\right)\lesssim e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j\approx 2^\ell$ ("scale ℓ"), have $H(j)-H(j-1)=\frac{1}{S_j}\simeq\frac{1}{2^\ell}$

So bound (a) time to change scales,
(b) "# visits to scales" $=(V(\ell),\ell\ge 1)$

(b) $V(\ell)=$ # visits to scale ℓ $=\#\{i: L_i=\ell\}$

Fact: Given that $V(\ell)\neq 0$, $V(\ell)$ dominated by sum of 2 Geom($\frac{1}{2}$) r.v.s; $\Longrightarrow \mathbb{P}(V(\ell)>k\,|\,V(\ell)>0)\lesssim 2^{-k/2}$.

Proof: visits to scale ℓ entail upcrossings of $[2^{\ell-1}, 2^{-\ell})$ or of $[2^{\ell+1}, 2^{\ell+2})$.
Both are hard since walk has non-positive drift. ∎

$W(\sigma)=\max(S_i, 0\le i<\sigma)$ $\quad H(\sigma)=\sum_{i=1}^{\sigma}\frac{1}{S_i}$ $\quad$ Aim: $\mathbb{P}\left(H(\sigma)\ge \frac{k}{1-p_1}W(\sigma)\right)\le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j\approx 2^\ell$ ("scale ℓ"), have $H(j)-H(j-1)=\frac{1}{S_j}\simeq\frac{1}{2^\ell}$

So bound (a) time to change scales,

(b) "# visits to scales" $=(V(\ell),\ell\ge 1)$

k'th scale visited

(a) $\underbrace{\tau_{k+1}-\tau_k}_{\text{time to change scale}}\sim(2^{L_k})^2$ so $\boxed{\mathbb{P}\left(H_{\tau_{k+1}}-H_{\tau_k}\ge\frac{x}{1-p}\cdot 2^\ell \,\middle|\, L_k=\ell\right)\le e^{-\delta x}}$

(b) $V(\ell)=$ # visits to scale ℓ $\quad\boxed{\mathbb{P}(V(\ell)>k\mid V(\ell)>0)\lesssim 2^{-k/2}}$

(a)+(b) $\Longrightarrow$ Total contribution of level ℓ to height is

$$\begin{cases}0 & \text{if } V(\ell)=0.\\ O(2^\ell) \text{ with exp. tails} & \text{if } V(\ell)>0.\end{cases}$$

So $H(\sigma)\lesssim 2^{\max(\ell:V(\ell)\neq 0)}\lesssim W(\sigma).$

Remarks

- Stronger results if add info. about tails of degrees.

Ex

- If $\mu(p) \leq 1$, $\mathrm{Var}(p) = v \in (0,\infty)$ then $\exists\, x_0$ s.t. $\forall x \geq x_0$,

$$\mathbb{P}\left(ht(T) \geq Kx|T|^{1/2}\right) \leq \exp(-vx^2)$$

- If in fact $\mu(p) = 1$ then $\forall n \geq 1$,

$$\mathbb{P}\left(ht(T) \geq Kx|T|^{1/2} \,\middle|\, |T| \geq n\right) \leq \exp(-vx^2)$$

- If $\exists\, \alpha \in (1,2]$, $M > 0$ s.t. $\sum_{j>i} p_j > \frac{M}{i^{\alpha}}$ $\forall i \geq 1$, then $\forall x \geq 1$,

$$\mathbb{P}\left(ht(T) \geq \frac{Kx}{M(\alpha-1)} \cdot |T|^{\frac{\alpha-1}{\alpha}}\right) \leq \exp(-x^{\alpha})$$

- If $\mu(p) \leq 1$, $\mathrm{var}(p) = \infty$ then $\forall \varepsilon > 0$ $\exists\, n_0$ s.t. $\forall x > 0$, $n \geq n_0$,

$$\mathbb{P}\left(ht(T) \geq x|T|^{1/2}, |T| \geq n\right) \leq \frac{x}{n^{1/2}} \exp(-x^2/\varepsilon)$$

Conjectures

- Conjecture: All this works even conditional on size of tree: $\mathbb{P}(ht(T) > A \cdot m \cdot wid(T) \mid \sigma = n) \leqslant \exp(-\delta m)$

  NB: Here should have $\delta = \delta(p_0, p_1)$

- Conjecture: Binary trees are the tallest.

  More specifically:

  Consider random trees $T_{\vec{n}}$ with a fixed degree seq $\vec{n} = (n_i, i \geq 0)$

  Here n_i = # nodes with i children. With $\sum n_i = \sigma$, then $\sum i n_i = \sigma - 1$

  Then to stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0 = n$, $n_1 = 0$, one should choose the seq. $(n, 0, n-1, 0, 0, 0, \dots)$

Part 2:

Binary trees stochastically maximize the height of a random node.

Conjecture: Binary trees are the tallest.

Consider random trees $T_{\vec{n}}$ with a fixed degree seq $\vec{n}=(n_i, i\geq 0)$.

Here n_i = # nodes of $\underbrace{\text{deg}}_{\text{\# children}}$ i.

To stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0=n$, $n_1=0$,

choose the seq. $bin(n)=(n,0,n-1,0,\ldots)$

"Evidence".

Proposition: Let $\vec{n}=(n,0,n_2,n_3,\ldots)$, let $(T_{\vec{n}}, V)$ be a random marked tree with degree sequence $\vec{n}$.

distinguished node

Let $bin(n)=(n,0,n-1,0,0,\ldots)$, let $(T_{bin(n)}, W)$ be a random marked binary tree with n leaves.

Then $height(V) \preceq_{st} height(W)$.

Main idea of proof: grow the paths from the roots to V and to W one step at a time, compare conditional stopping probabilities.

(More details forthcoming.)

Warm-up:

#binary trees, n leaves $= \frac{1}{2n-1}\binom{2n-1}{n}$

#trees with degree seq. $\vec{n}$. With $|\vec{n}| := \sum n_i$ then get

$$\frac{1}{|\vec{n}|}\binom{|\vec{n}|}{n_i, i \geq 0} = \frac{1}{|\vec{n}|}\frac{|\vec{n}|!}{n_0!\, n_1! \cdots}$$

$= \frac{1}{|\vec{n}|} \cdot$ # lattice walks, n_i steps of size $i-1$,

#binary forests, n leaves, k connected components $= \frac{k}{2n-k}\binom{2n-k}{n}$

#forests with degree seq. $\vec{n}$. With k trees, then get $\frac{k}{|\vec{n}|}\binom{|\vec{n}|}{n_i, i \geq 0}$

$= \frac{k}{|\vec{n}|} \cdot$ # lattice walks, n_i steps of size $i-1$,

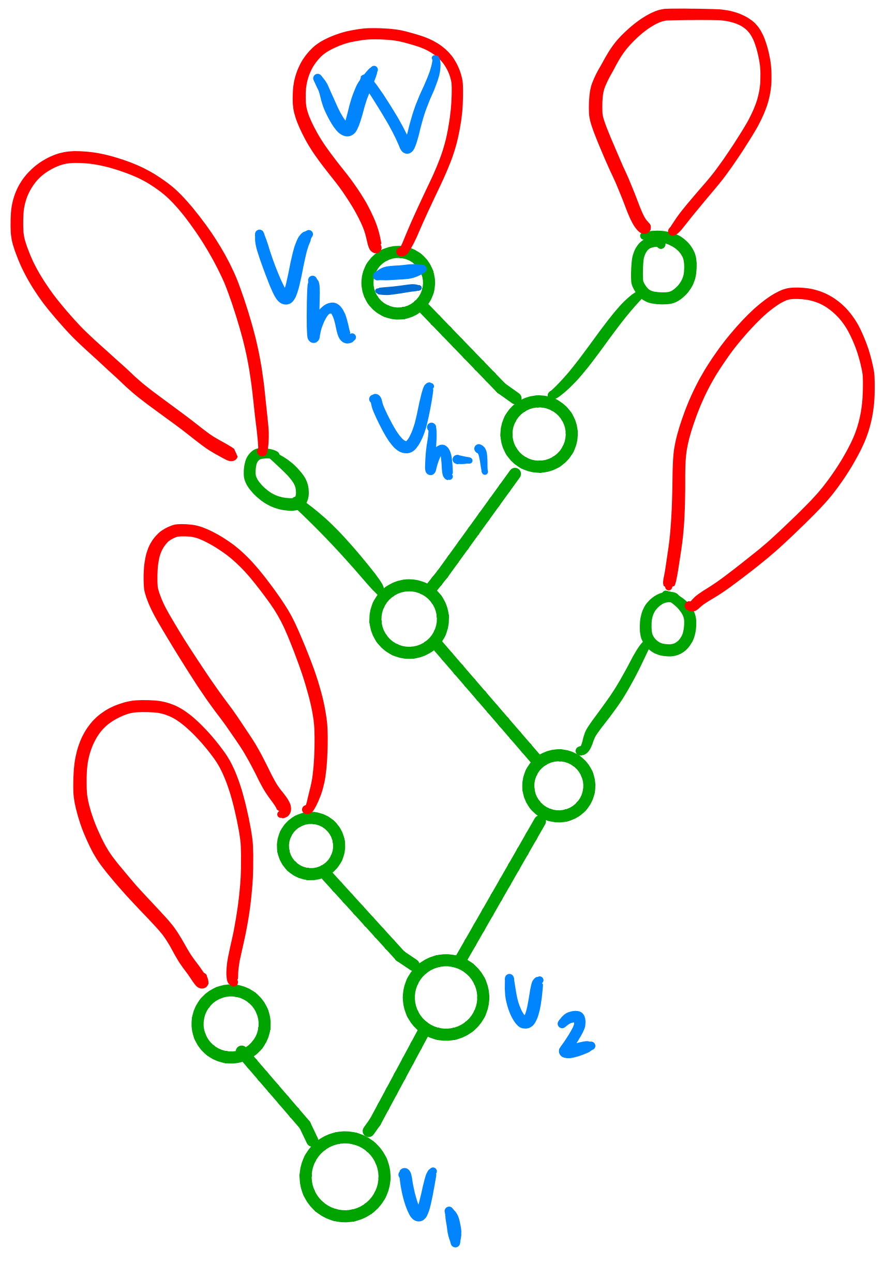
#marked forests with degree seq. $\vec{n}$, mark in last tree

distinguished vertex

$= |\vec{n}| \cdot \frac{1}{k} \cdot$ # forests with degree seq. $\vec{n}$

$= \binom{|\vec{n}|}{n_i, i \geq 0} =$ # lattice walks, n_i steps of size $i-1$

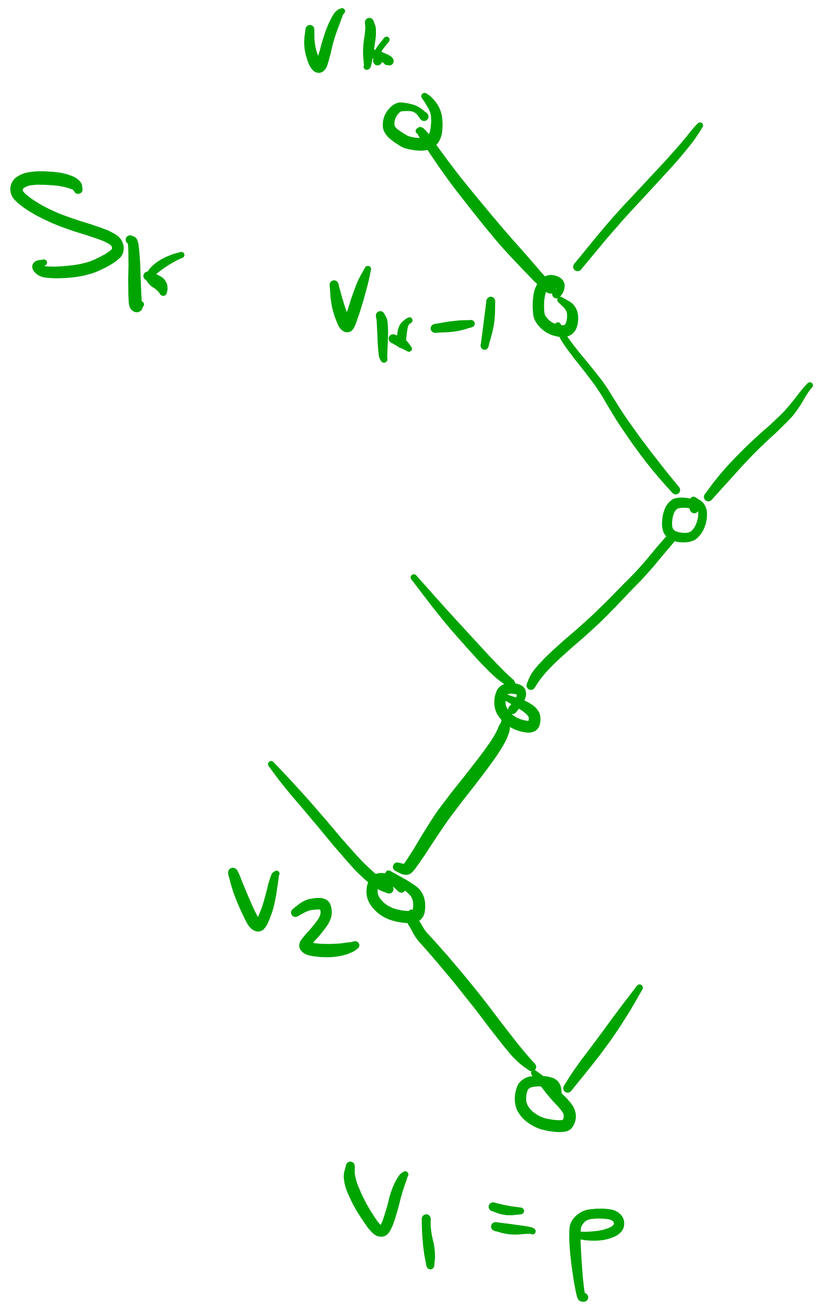
Trunks of trees



Let (T, W) be a random marked binary tree, n leaves.

Trunk = path from root to marked vertex,

together with children of path vertices

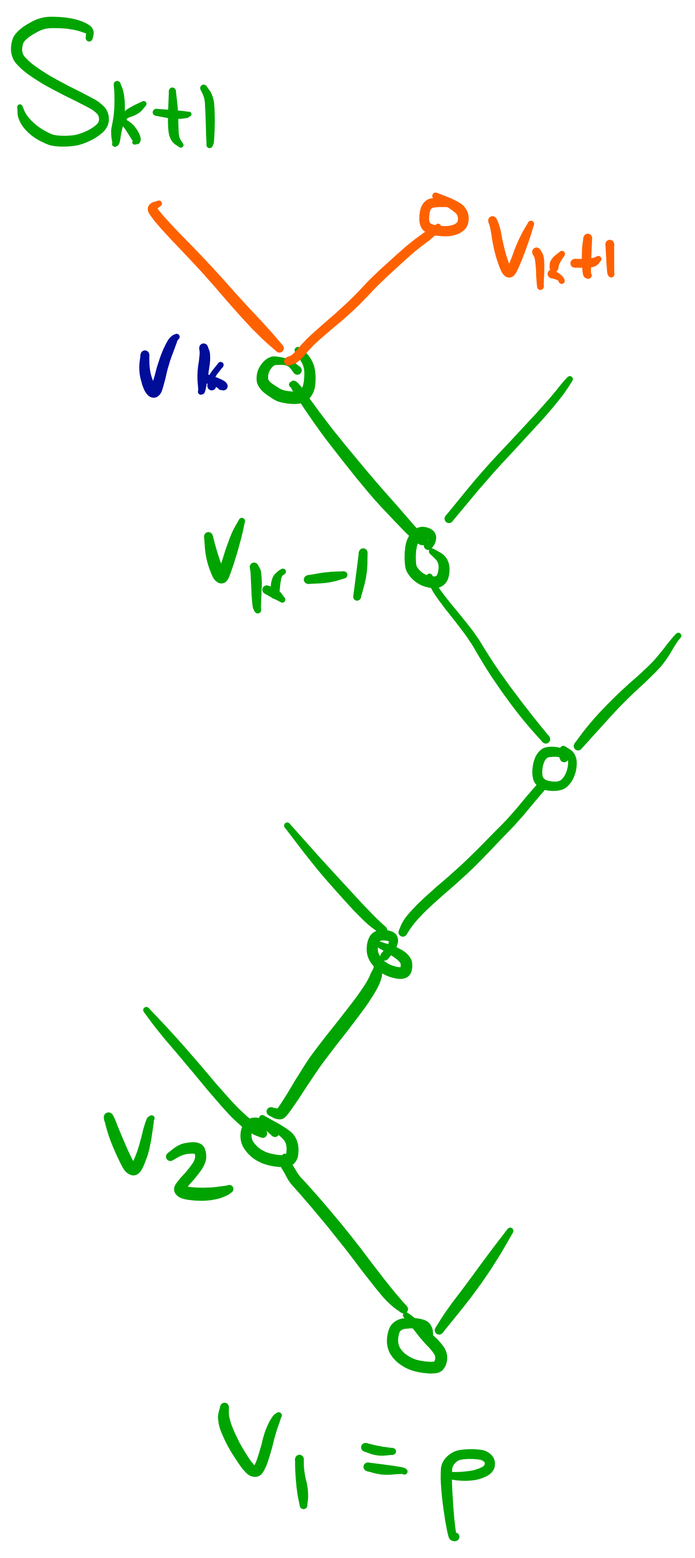


$\#$ Marked binary trees with n leaves, trunk containing S_k,

(marked node in subtree rooted at v_k)

= $\#$ binary forests with n leaves, k trees, mark in last tree = $\binom{2n-k}{n}$

$\#$ Marked binary trees with n leaves, trunk containing S_{k+1}, = $\binom{2n-k-1}{n}$

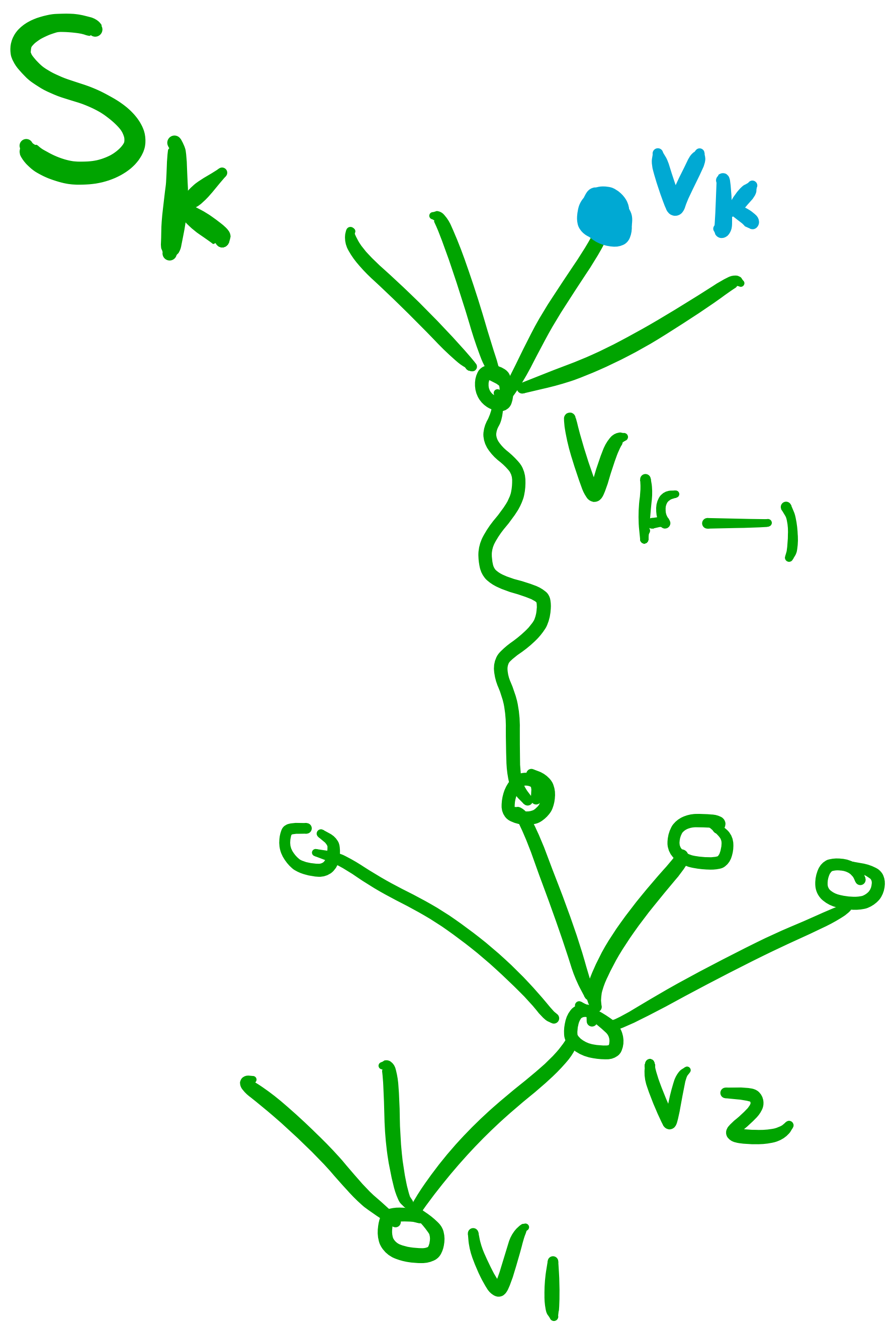


Ratio is $\frac{(2n-k-1)!}{n!(n-k-1)!} \cdot \frac{n!\,(n-k)!}{(2n-k)!} = \frac{n-k}{2n-k}$

Two possible choices for v_{k+1} (left or right)

So $\mathbb{P}(W = v_k \mid \text{Trunk contains } S_k) = 1 - 2 \cdot \frac{n-k}{2n-k} = \frac{k}{2n-k}$

$= 1/\mathbb{E}(\text{Size of subtree above } v_k \mid T \text{ contains } S_k)$



Proposition

Let (T,V) be a random marked tree with degree sequence $\vec{n}$. Fix k and S_k with $\mathbb{P}(\text{Trunk contains } S_k) > 0$.

Then $\mathbb{P}(V = v_k \mid \text{Trunk contains } S_k)$

$$= 1/\mathbb{E}(\text{Size of subtree above } v_k \mid T \text{ contains } S_k)$$

$$= \frac{\overbrace{1 + \sum_{i=1}^{k-1}(\deg(v_i)-1)}^{\#\text{ subtrees hanging from trunk.}}}{\underbrace{|\vec{n}| - (k-1)}_{\#\text{ vertices in these subtrees}}}$$

NB: • With $\vec{n} = (n, 0, n_2, n_3, \dots)$ we have $|\vec{n}| \le 2n-1$

• Total # subtrees hanging from trunk is $1 + \sum_{i=1}^{k-1}(\deg(v_i)-1) \geq k$,

so Proposition implies

$$\mathbb{P}(V = v_k \mid T \text{ contains } S_k) \geq \frac{k}{2n-k}$$

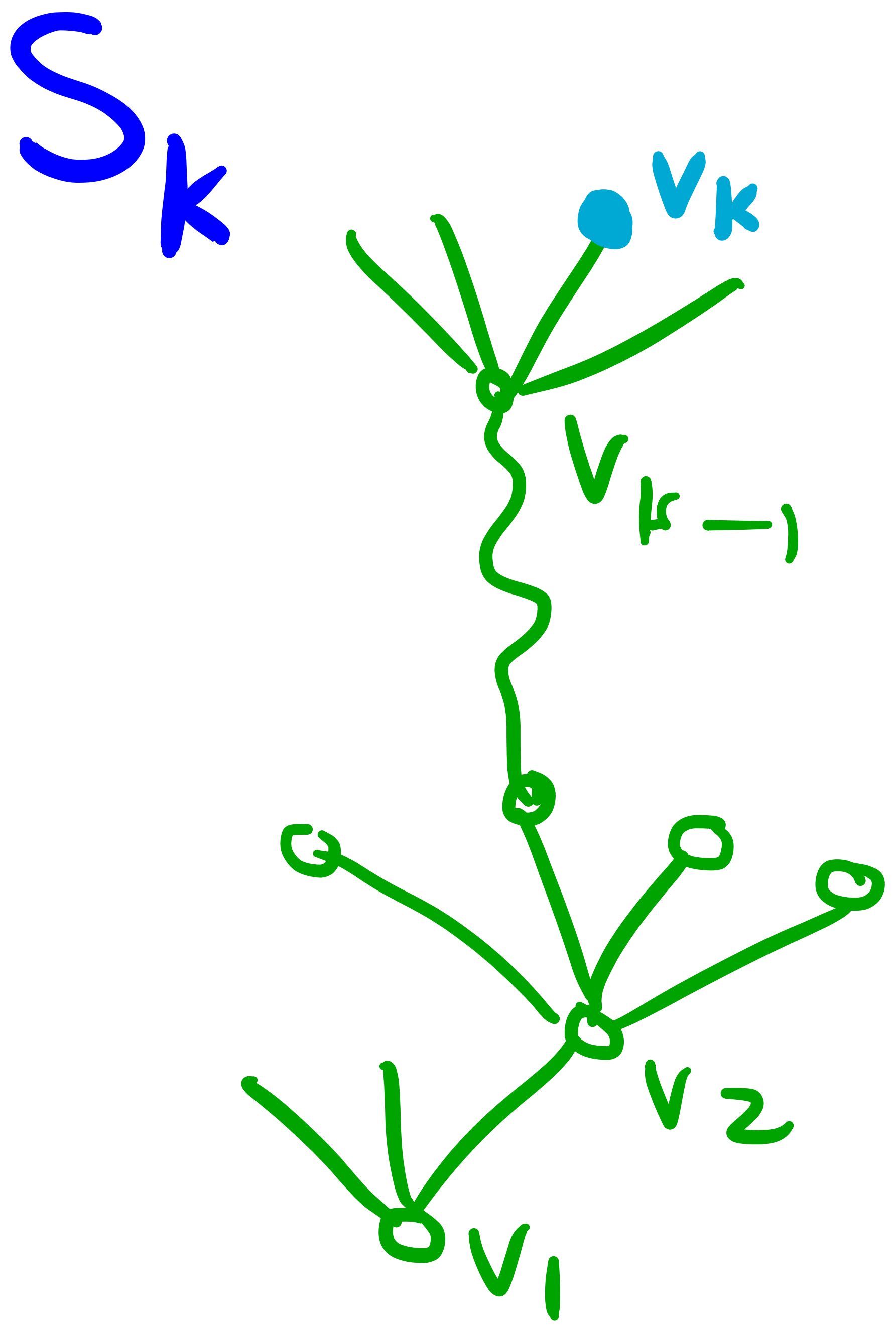
Proposition: Let $\vec{n} = (n, 0, n_2, n_3, \dots)$, let $(T_{\vec{n}}, V)$ be a random marked tree with degree sequence $\vec{n}$.

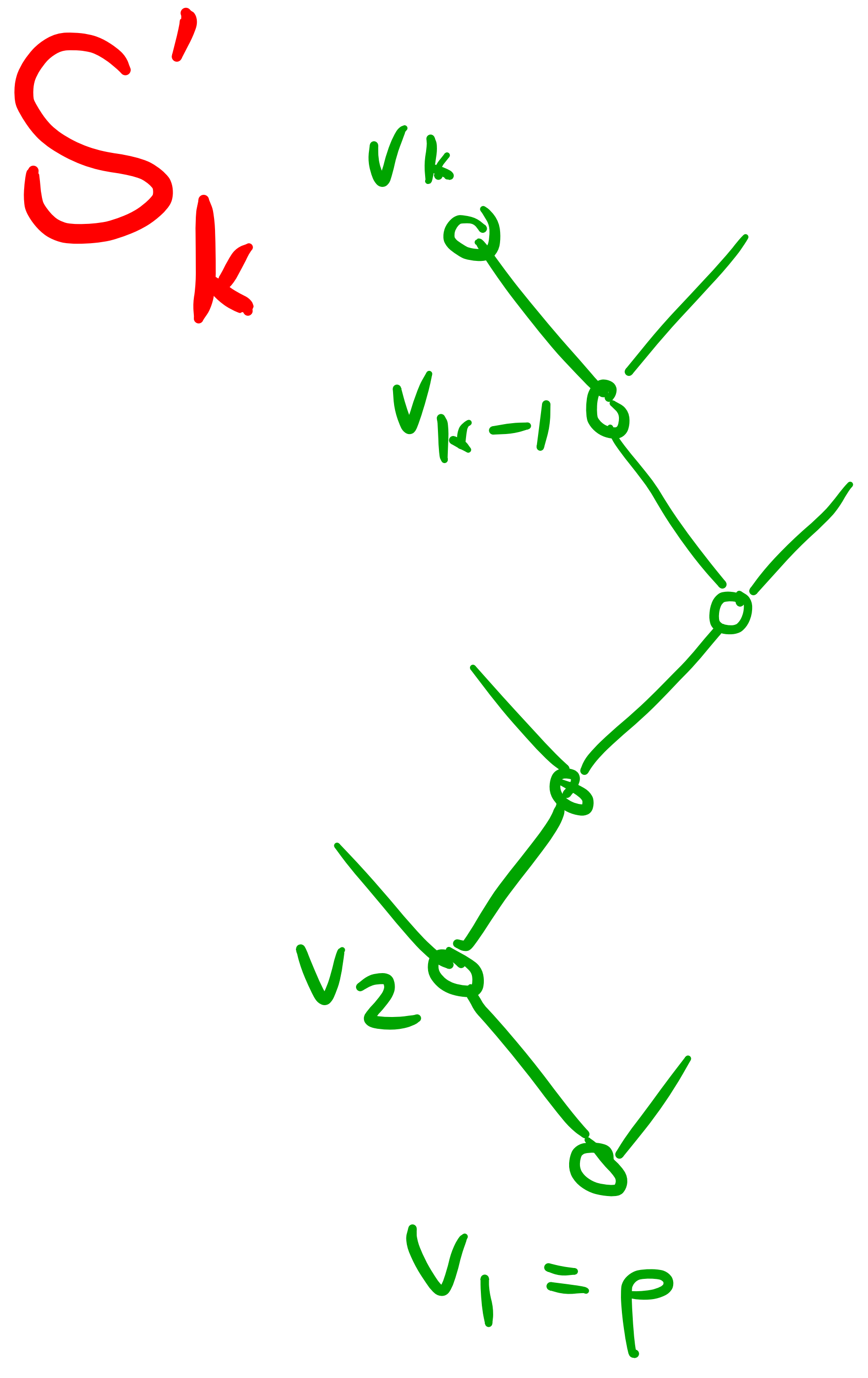
distinguished node

Let $\mathrm{bin}(n) = (n, 0, n-1, 0, 0, \dots)$, let $(T_{\mathrm{bin}(n)}, W)$ be a random marked binary tree with n leaves.

Then $\mathrm{height}(V) \preccurlyeq_{st} \mathrm{height}(W)$.

Proof: Fix S_k, S'_k possible "sub-trunks" for $T_{\vec{n}}$, $T_{\mathrm{bin}(n)}$.





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Then $\mathrm{height}(V) \preceq_{st} \mathrm{height}(W)$.

Proof: Fix S_k, S'_k possible "sub-trunks" for $T_{\vec{n}}$, $T_{\mathrm{bin}(n)}$.

In marked binary tree $\mathbb{P}(W = v_k \mid \text{Trunk contains } S'_k) = \dfrac{k}{2n-k}$

In marked tree with degree sequence $\vec{n}$, $\mathbb{P}(V = v_k \mid \text{Trunk contains } S_k) \geq \dfrac{k}{2n-k}$

So can couple step-by-step constructions of trunks so that at each step, the construction of marked tree with degree sequence $\vec{n}$ is more likely to halt (if it has not already done so). $\blacksquare$

This sampling technique has other uses.

Theorem: There exists $c > 0$ such that the following holds. Fix any degree sequence $\vec{n} = (n_0, n_1, n_2, \ldots)$, let $(T_{\vec{n}}, V)$ be a random marked tree with degree sequence $\vec{n}$. Write $\mathrm{var}(\vec{n}) = \sum_{i>0} i(i-1)\frac{n_i}{|\vec{n}|}$. Then for all $x > 0$,

$$\mathbb{P}(\mathrm{ht}(V) \geq x|\vec{n}|^{1/2}) \leq \exp(-cx^2\mathrm{var}(\vec{n})), \text{ and } \mathbb{E}[\mathrm{ht}(V)] \leq \frac{1}{2c\,\mathrm{var}(\vec{n})} \cdot |\vec{n}|^{1/2}.$$

Corollary: $\mathbb{E}[\mathrm{wid}(T_{\vec{n}})] \geq \frac{c\,\mathrm{var}(\vec{n})}{8} \cdot |\vec{n}|^{1/2}$

This sampling technique has other uses.

Theorem: There exists $c>0$ such that the following holds. Fix any degree sequence $\vec{n}=(n_0,n_1,n_2,\dots)$, let $(T_{\vec{n}}, V)$ be a random marked tree with degree sequence $\vec{n}$. Write $\mathrm{var}(\vec{n})=\sum_{i>0} i(i-1)\frac{n_i}{|\vec{n}|}$. Then for all $x>0$,

$$\mathbb{P}\left(ht(V)\geq x|\vec{n}|^{1/2}\right)\leq \exp\left(-c x^2 \mathrm{var}(\vec{n})\right), \text{ and } \mathbb{E}[ht(V)]\leq \frac{1}{2c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}.$$

Corollary: $\mathbb{E}[wid(T_{\vec{n}})]\geq \frac{c\,\mathrm{var}(\vec{n})}{8}\cdot|\vec{n}|^{1/2}$

Proof: By Markov's inequality,

$$\mathbb{P}\left[ht(V)\geq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}\right]\leq \mathbb{P}\left[ht(V)\geq 4\mathbb{E}[ht(V)]\right]\leq \frac{1}{4}$$

Also $\mathbb{P}\left[ht(V)\geq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}\right]\geq \frac{1}{2}\cdot\mathbb{P}\left[\#\left\{u\in T_{\vec{n}}: ht(u)\geq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}\right\}\geq \frac{\vec{n}}{2}\right]$

So $\mathbb{P}\left[\#\left\{u\in T_{\vec{n}}: ht(u)\geq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}\right\}\geq \frac{|\vec{n}|}{2}\right]\leq \frac{1}{2}$

This sampling technique has other uses.

Theorem: There exists $c>0$ such that the following holds. Fix any degree sequence $\vec{n}=(n_0,n_1,n_2,\dots)$, let $(T_{\vec{n}}, V)$ be a random marked tree with degree sequence $\vec{n}$. Write $\mathrm{var}(\vec{n})=\sum_{i>0} i(i-1)\frac{n_i}{|\vec{n}|}$. Then for all $x>0$,

$$\mathbb{P}(\mathrm{ht}(V)\geq x|\vec{n}|^{1/2}) \leq \exp(-c x^2 \mathrm{var}(\vec{n})), \text{ and } \mathbb{E}[\mathrm{ht}(V)] \leq \frac{1}{2c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}.$$

Corollary: $\mathbb{E}[\mathrm{wid}(T_{\vec{n}})] \geq \frac{c\,\mathrm{var}(\vec{n})}{8}\cdot|\vec{n}|^{1/2}$

Proof (continued): $\mathbb{P}\left[\#\left\{u\in T_{\vec{n}} : \mathrm{ht}(u)\geq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}\right\} \geq \frac{|\vec{n}|}{2}\right] \leq \frac{1}{2}$

But if $\exists \geq \frac{|\vec{n}|}{2}$ nodes at height $\leq \frac{2}{c\,\mathrm{var}(\vec{n})}\cdot|\vec{n}|^{1/2}$

then $\mathrm{wid}(T_{\vec{n}}) \geq \frac{c\,\mathrm{var}(\vec{n})}{4}\cdot|\vec{n}|^{1/2}$ by the pigeonhole principle.

So $\mathbb{E}[\mathrm{wid}(T_{\vec{n}})] \geq \frac{c\,\mathrm{var}(\vec{n})}{4}\cdot|\vec{n}|^{1/2}\cdot\frac{1}{2}$ ∎

Results

2,4 Tight;
1,3,5 off by $(\log n)^{1/2}$ factor.

Theorem 1: If $\mathbb{E}X=1$, $\mathbb{E}X^2=\infty$ then $\mathrm{ht}(T_\omega^{(n)})/(n\log n)^{1/2} \xrightarrow{P} 0$

Theorem 2: If $\mathbb{E}X=1$, $\mathbb{E}X^2=\infty$ then $\mathrm{wid}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} \infty$

Theorem 3: If $\mathbb{E}X<1$ then $\mathrm{ht}(T_\omega^{(n)})/(n\log n)^{1/2} \xrightarrow{P} 0$

Theorem 4: If $\mathbb{E}X<1$ then $\mathrm{wid}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} \infty$

Theorem 5: $\mathrm{ht}(T_\omega^{(n)})/(n\log n)^{1/2}$ has uniform subgaussian tails

Conjectures of Svante Janson from "Simply-generated trees, conditioned Galton-Watson trees, random allocations and condensation", Prob. Surveys Vol 9 (2012), pages 103-252

Conjecture 21.5: If $\mathbb{E}X=1$, $\mathbb{E}X^2=\infty$ then $\mathrm{ht}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} 0$

Conjecture 21.6: If $\mathbb{E}X=1$, $\mathbb{E}X^2=\infty$ then $\mathrm{wid}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} \infty$

Problem 21.7: Does $\mathbb{E}X<1$ imply that $\mathrm{ht}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} 0$?

Problem 21.8: Does $\mathbb{E}X<1$ imply that $\mathrm{wid}(T_\omega^{(n)})/n^{1/2} \xrightarrow{P} \infty$?

Problem 21.9: Does $\mathrm{ht}(T_\omega^{(n)})/n^{1/2}$ have uniform subgaussian tails?

Can pass results to simply generated trees by "averaging over the degree sequence".

Svante Janson: Simply Generated Trees, conditioned Galton-Watson Trees, Random Allocations and Condensation.

Prob. Surveys Vol 9 (2012) 103-252

Conjecture 21.5. *If $\nu = 1$ and $\sigma^2 = \infty$, then $H(\mathcal{T}_n)/\sqrt{n} \xrightarrow{p} 0$.* $\sqrt{n \log n}$ ✓

Conjecture 21.6. *If $\nu = 1$ and $\sigma^2 = \infty$, then $W(\mathcal{T}_n)/\sqrt{n} \xrightarrow{p} \infty$.* ← ✓

Problem 21.7. *Does $\nu < 1$ imply that $H(\mathcal{T}_n)/\sqrt{n} \xrightarrow{p} 0$?* $\sqrt{n \log n}$ ✓

Problem 21.8. *Does $\nu < 1$ imply that $W(\mathcal{T}_n)/\sqrt{n} \xrightarrow{p} \infty$?* ← ✓

Furthermore, still in the case $\nu \geqslant 1$, $\sigma^2 < \infty$, Addario-Berry, Devroye and Janson [1] have shown sub-Gaussian tail estimates for the height and width

$$\mathbb{P}(H(\mathcal{T}_n) \geqslant x\sqrt{n}) \leqslant Ce^{-cx^2}, \tag{21.12}$$

$$\mathbb{P}(W(\mathcal{T}_n) \geqslant x\sqrt{n}) \leqslant Ce^{-cx^2}, \tag{21.13}$$

uniformly in all $x \geqslant 0$ and $n \geqslant 1$ (with some positive constants C and c depending on $\boldsymbol{\pi}$ and thus on $\mathbf{w}$). In view of (21.11), we cannot expect (21.13) to hold when $\sigma^2 = \infty$ (or when $\nu < 1$), but we see no reason why (21.12) cannot hold; (21.10) suggests that $H(\mathcal{T}_n)$ typically is smaller when $\sigma^2 = \infty$.

Problem 21.9. *Does (21.12) hold for any weight sequence $\mathbf{w}$ (with C and c depending on $\mathbf{w}$, but not on x or n)?* Up to $\sqrt{\log n}$ factor ✓

Thank you!

- Claimed theorem with dependence only on p_1 proved it with dependence on $p = \max p_i$.

Fix: requires more careful "disperson" bound for our setting.

(Idea: If subcritical then $p_{max} = p_0$ or p_1; if p_0 close to 1 then either very subcritical or make large jumps.)